Efficiency in Matching Markets: application cost in school choice*

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Abstract

In our study of the impact of application costs on two-sided matching markets, we consider the student payoff to be a lexicographic function that takes into account both the assigned school and the associated fees to be paid. Using the deferred acceptance algorithm (DA), we identify a cost profile that leads to a Pareto-efficient equilibrium outcome, which dominates the outcome without costs. Additionally, we analyze the Boston and Top Trading Cycle mechanisms and conclude that application costs do not have a positive impact. We identify the condition that costs must respect for Pareto improvement in DA. The consequence is the impossibility of treating students equally. We also find that strategic behavior by schools can prevent Pareto improvements, as they prioritize their own interests.

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1 Introduction

Schools play a vital role in shaping students' career prospects and are instrumental in promoting equal opportunities. To support students who demonstrate financial need, governments, associations, or schools can offer scholarships that cover various expenses, including tuition fees, study abroad costs, and application fees. Although application fees are generally low, they can pose a significant obstacle for students from disadvantaged backgrounds. As a result, such students may limit the number of schools they apply to, which reduces their chances of being accepted. Additionally, they may avoid applying to highly selective schools due to the high costs involved, which can further limit their opportunities. Given the impact of application costs on students' educational decisions, it raises questions about how such costs influence their application strategies and whether there is a cost design that can facilitate effective matching. This paper aims to contribute to addressing these questions.

While matching mechanisms are typically administered centrally to assign students to schools, it is the schools themselves who set the application fees. These fees serve two main purposes. Firstly, they help to ensure that the student's application is genuine. Without a fee, some students may apply to numerous programs they have little interest in, leading to greater competition for available seats.[1] Secondly, the fees help to cover the administrative costs associated

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[1]This is known as the congestion phenomenon, and it can have negative consequences for students.

with evaluating each application. For schools that receive a large number of applications, this can be a time-consuming and costly process. The application fees can vary between schools, but on average, they tend to be around $50 per application.[2]

To reduce inequality in the application process, some students are granted exemption from application fees, which facilitates their acceptance into a school. By doing so, students from disadvantaged backgrounds can apply to a wider range of programs, including more selective ones. There are several methods by which students can benefit from college application fee waivers. Typically, the student must complete a separate application process, and based on several criteria, the application fees can be waived. These criteria may include family income, enrollment in a government program for low-income students, living in federally subsidized public housing, being a ward of the state, or being an orphan, among others.[3] Many schools, such as the University of Pittsburgh, provide an application fee waiver if students visit the school, as it demonstrates the student's interest in the school. International students may also receive aid or have their application fees waived by schools, as the objective of these policies is to encourage exchanges between different cultures. The criteria for setting application fees are generally designed to meet social objectives.

The main goal of this paper is, therefore, to study the impact of costs on students' application strategies. He and Magnac (2022) empirically analyzes the effects of application fee implementation. To the best of our knowledge, there is no theoretical analysis of the impact of costs on strategy and property, such as efficiency. We investigate a cost design that leads to a Pareto improvement in assignments under the well-known deferred acceptance mechanism (DA hereafter) introduced by Gale and Shapley (1962). For this purpose, we set the costs to reduce the number of student applications, which then modified their strategies.[4] In our approach, we consider that the application costs are very low. The objective is not to impact the order of students' preference for schools. In other words, a school will not be less valued by a student because of its application cost. In our first result (Proposition 1), we show that costs may prevent the dominance of the truthful revelation strategy. Thus, it is essential to note that even a low application cost impacts the strategies. Knowing this, we can consider costs that lead to a strategy that is not dominated and that results in an equilibrium producing a Pareto-efficient outcome and that Pareto dominates DA. We state this result in our Theorem 1. To reduce the number of applications to be reviewed by the schools and the associated costs, we identify a cost profile leading to an equilibrium such that the outcome is Pareto-efficient and students apply to no more than two schools. However, the cost profile must be designed accurately (Proposition 2).

In a school choice problem, efficiency is defined with respect to students' preferences only. Efficiency is achieved when no student's assignment can be improved without deteriorating the

[2]However, fees can differ depending on factors such as the testing requirements of a particular program or the number of applications received by the college. Some schools, such as Stanford University, Duke University, and Columbia University, have particularly high fees, ranging from $80 to $90.

[3]This process can be automated, as with the College Board's SAT fee waiver, which provides four college application fee waivers, or students can apply online, and colleges can waive the application fee independently.

[4]We aim to reduce congestion in the school market by implementing application costs.

assignment of any others. Another important criterion in matching is stability. Stability[5] is crucial for the long-term sustainability of a market, and it also guarantees a sense of fairness. However, efficiency and stability are incompatible (Roth, 1982). To address this issue, Kesten (2010) propose an algorithm called the Efficiency-Adjusted Deferred Acceptance Mechanism (EADAM), which modifies the student preferences to achieve efficient matching. The EADAM identifies the applications that create inefficiency in the DA algorithm and modifies the students' preferences to reach a Pareto-efficient matching. Since the DA outcome is the stable assignment preferred by the students, it is possible to consider an assignment that is both efficient and Pareto-dominant to the DA outcome. However, Kesten (2010) does not consider equilibria.

Many school systems worldwide use centralized[6] mechanisms to assign students to schools. In the school choice model, the agents' strategic choices are crucial in the design of a matching mechanism. The literature focuses on Nash equilibria in different mechanisms. Abdulkadiroğlu and Sönmez (2003) analyze the weaknesses of the mechanisms used until recently. In this paper, we show that application costs cannot lead to a Pareto improvement in two mechanisms often mentioned in the literature, namely the Boston and the Top Trading Cycles (TTC hereafter). For the Boston mechanism, low costs do not prevent student deviations. Thus, if no Pareto-efficient equilibrium exists without cost, creating a Pareto-efficient equilibrium with low costs is impossible (Proposition 4). For TTC, [7] the costs have a negative impact on students without generating an improvement (Proposition 5). Thus, in these two mechanisms, cost implementation does not impact the equilibria.

We also investigate the impact of costs under DA and identify a necessary condition for achieving a Pareto improvement. To achieve efficiency, we need to consider both student preferences and school priorities when setting costs. Our analysis reveals that all students cannot be treated equally, and some may have to pay an application fee while others don't (Corollary 1). Therefore, costs must depend on both the schools and the students. If costs are only based on schools, they may hinder the attainment of an efficient outcome. This has significant political implications as it requires a central planner to implement costs or coordinate with schools to establish fees for achieving a Pareto-efficient outcome. Finally, we provide a key result showing the cost profiles that prevent the existence of Pareto-efficient equilibria (Theorem 2).

Lastly, we examine the significance of cost design and explore the strategic behavior of schools in setting application costs. Our analysis reveals that schools have a tendency to set a positive cost for students assigned in the stable student-optimal matching and a zero cost for those who will not be admitted to the school. This finding contradicts the common belief that schools set zero costs to attract the best students and positive costs to deter others from applying. Unfortunately, this strategy employed by schools undermines the possibility of achieving Pareto-efficient equilibria. It is noteworthy that a Pareto improvement for students may lead to a worsening of the situation for schools. Hence, by preventing Pareto improvements, schools ensure an equilibrium outcome at least as desirable as the one obtained in DA without costs. Nevertheless, we have not taken into account the reviewing cost for schools in our analysis. Therefore, the

[5]DA suggests the stable assignment preferred by students in the school choice context.

[6]In a centralized market, agents submit information about their preferences to a market operator. The market operator uses the collected preference information to match agents.

[7]It is known that the TTC yields a Pareto-efficient assignment and that it is strategy-proof (Abdulkadiroğlu and Sönmez, 2003).

application fee waiver may serve the schools' interests and harm the students. It is crucial to incorporate the reviewing cost into the centralized mechanisms to address this issue.

Our research is also relevant to the extensive literature on constrained matching, where the constraint may limit the number of schools that students can report on their preference lists. For instance, Haeringer and Klijn (2009) investigate the Nash equilibria under three popular mechanisms - DA, Boston, and TTC - when constrained by the number of schools that students can report. As the quota system eliminates the existence of dominant strategies for DA and TTC mechanisms, the authors focus on Nash equilibria of the quota games. In contrast, our study introduces a constraint on the schools that students report, and we examine the impact of application costs on the set of Nash equilibria. Moreover, our analysis considers rational self-constraint by students at equilibrium, while Haeringer and Klijn (2009) assume an exogenous constraint.

The organization of the rest of the paper is as follows. In Section 2, we recall the model of school choice and model strategies and costs. In Section 3, we investigate the existence of Nash equilibrium resulting in a Pareto-efficient matching in the presence of costs. In Section 4, we study the impact of costs on Boston and TTC mechanisms. In Section 5, we consider the conditions of the application's cost distributions. In Section 6, we discuss our results by investigating the schools' strategies. In Section 7, we conclude. All the proofs are collected in the appendix.

2 Model

2.1 School choice

Following Abdulkadiroğlu and Sönmez (2003), we define a *school choice problem* by a set of schools and a set of students, where each student has to be assigned a seat at not more than one of the schools. Each student is assumed to have strict preferences over the set of schools and the possibility of remaining unassigned. Each school is endowed with a priority[8] ordering over the students and a fixed capacity of seats. Let $\mathcal{I}$ be the nonempty finite set of *students*. A generic element in $\mathcal{I}$ is denoted by i. Let $\mathcal{S}$ be the nonempty finite set of *schools*. A generic element in $\mathcal{S}$ is denoted by s. For each $s \in \mathcal{S}$, we denote by $q_s \in \mathbb{N}^*$ the *number of available seats* at school s. We denote by $q = (q_{s_1}, ..., q_{s_n})$ the *capacity vector* of schools. Let denote the *strict preference profile* by $P = (P_i)_{i \in \mathcal{I}}$ which is a vector of linear orders (complete, transitive, and antisymmetric relations) of students over the set of schools, such that P_i denotes the preference of student i over $\mathcal{S} \cup \{i\}$. We denote by sR_it if student i *weakly prefers* school s to school t meaning sP_it or $s = t$. A school s is *acceptable* to i if sR_ii. Being assigned to oneself is interpreted as *not being assigned* to any school.

Let us denote the *strict priority structure* by $\succ = (\succ_s)_{s \in \mathcal{S}}$, which is a priority order (a complete, transitive, and antisymmetric relation) for schools over the set of students such that $\succ_s$ denotes the priority of school s over $\mathcal{I} \cup \emptyset$. Being assigned to $\emptyset$ is interpreted as having *no students*

[8] There are key differences between priorities and preferences. The most important is that preferences can change while priorities are fixed. The fact that preferences can be changed evokes strategic considerations, while priorities do not. Finally, preferences represent the Pareto efficiency criterion, while priorities are irrelevant from a welfare perspective.

assigned to the school. We denote by $i \succeq_s j$ if school s has a *weak priority* for i compared to j. For any $s \in \mathcal{S}$ and $i, j \in \mathcal{I}$, with $i \neq j$, let $i \succ_s j$ if and only if $i \succeq_s j$ and not $j \succeq_s i$. In this article, we will generally consider the priority orders to be strict.

An assignment is a function $\mu : \mathcal{I} \to \mathcal{S} \cup \mathcal{I}$ that maps each student to either a school or the set of unassigned students. We denote $\mu(i)$ as the assignment of student i under matching μ. Formally an assignment is a function μ satisfying:

(i) $\forall i \in \mathcal{I}, \mu(i) \in \mathcal{S} \cup \{i\}$,

(ii) $\mu^{-1}(s) \in 2^{\mathcal{I}}$,

(iii) $\forall s \in \mathcal{S}, |\mu^{-1}(s)| \leq q_s$,

(iv) $\mu(i) = \{s\}$ if and only if $i \in \mu^{-1}(s) \cup \{i\}$

For simplicity, we write $\mu(i) = s$ instead of $\mu(i) = \{s\}$. If $\mu(i) = \{i\}$ we say that i is unassigned at μ. A school choice matching problem is a tuple $(\mathcal{I}, \mathcal{S}, P, \succ, q)$. Because $\mathcal{I}$ and $\mathcal{S}$ are fixed a school choice matching problem is given by $(P, \succ, q)$.

We now introduce some properties often used in the matching literature.
A matching μ *violates student i's priority* if there is $s \in \mathcal{S}$ such that:

- $sP_i\mu(i)$ and either $i \succ_s j$ for some $j \in \mu^{-1}(s)$ or $|\mu^{-1}(s)| < q_s$.[9]

A matching μ is *individually rational* if there is $\mu(i) \in \mathcal{S} \cup \{i\}$ such that:

- $\forall i \in \mathcal{I}, \mu(i) R_i i$.

A matching μ is *stable* if it does not violate any student's priority at any school and it is individually rational.
Another desirable property for a matching is Pareto-efficiency. We say that μ' *Pareto dominates* μ if:

- $\forall i \in \mathcal{I}, \mu'(i) R_i \mu(i)$.

A matching μ is *Pareto-efficient* if it is not Pareto dominated by any other matching.

2.2 Cost and strategy

In this article, we consider the implementation of application fees that impact students' payoff. Students can act strategically by changing their reported preferences to maximize their payoffs. Let $\mathcal{Q} = (Q_{i_1}, ..., Q_{i_n})$ be a *strategy profile* such that Q_i is the ordered list of school preferences of student i over the set of schools. For instance, we denote the strategy of i by $Q_i : s_1, s_2, s_3, i, s_4$. This is interpreted as in its strategy, i ranks s_1 higher than s_2 and s_2 higher than s_3. i prefers not to be assigned to a school rather than being assigned to s_4. We can simplify this as $Q_i : s_1, s_2, s_3, i$. We use the notation $s \notin Q_i$ to indicate that student i is not applying to school s under strategy Q_i.

A rule φ is a function that associates an assignment to every problem $(P, \succ, q)$. A rule φ is *strategy-proof* if no student can ever benefit by unilaterally misrepresenting their preferences,

[9] See Balinski and Sönmez (1999). In the school choice context, the idea is that if a school has not fulfilled its capacity, then a student can be admitted.

meaning that for any P, i and $Q_i \neq P_i$ we have that for any $\mathcal{Q}_{-i}$:

- $\varphi_i((P_i, \mathcal{Q}_{-i}), \succ, q) R_i \varphi_i((Q_i, \mathcal{Q}_{-i}), \succ, q)$.

In the following, we denote by $\mathcal{Q}_{-i}$ the strategy profile of other students.

A strategy profile $\mathcal{Q}$ is an *equilibrium* if for every $i \in \mathcal{I}, Q_i$ is player i's best response to the strategies $\mathcal{Q}_{-i}$ of the other players. Formally for every $i \in \mathcal{I}$, there is no strategy Q'_i such that $Q'_i \neq Q_i$ and we have:

- $\varphi_i((Q'_i, \mathcal{Q}_{-i}), \succ, q) P_i \varphi((Q_i, \mathcal{Q}_{-i}), \succ, q)$.

Let us now introduce the application costs. The idea is to consider very low application costs such that the implementation of application fees does not interchange schools in the strict preferences of the students. Considering low application cost is relevant because being admitted to a school that the student strictly prefers over another is more important than an application cost. In this article, we consider that the costs depend on both the school and the student.[10] The costs are either strictly positive or zero. We denote by C the cost profile, which can be represented by a matrix of dimension $(|\mathcal{I}|, |\mathcal{S}|)$. For each school $s \in \mathcal{S}$, we denote by $c_{i,s}$ the coefficient of the matrix for the application cost of student i to school s. If the application cost for student i to school s is null, then $c_{i,s} = 0$, and if the cost is positive $c_{i,s} = 1$. Let c_{Q_i} denote the sum of the application costs to be paid by i when i uses the strategy Q_i under C.

We use a lexicographic criterion to determine each student's payoff $(\mu(i), c_{Q_i})$. The primary criterion is the school to which the student is assigned. The second criterion is the cost associated with the applications that constitute its strategy. Thus, the student will choose the least costly strategy for an identical assignment. We have $(\mu(i), c_{Q_i}) \gtrdot (\mu(i)', c'_{Q_i})$ if and only if:

- $\mu(i) P_i \mu(i)'$ or $\mu(i) = \mu(i)'$ and $c_{Q_i} < c'_{Q_i}$.

We say that a student i is negatively (resp. positively) *impacted* if, by comparing two situations, the payoff of student i is lower (resp. higher) in the new situation.

We use the interrupter concept presented in Kesten (2010) that we now define. An *interrupter* is characterized by a student-school pair (i, s) such that i generates a rejection chain at step k, with $k \geq 1$ of DA, by applying to school s and is rejected at a later step[11] of s such that $\mu(i) \neq s$. Kesten (2010) establish that by neutralizing[12] certain interrupters, the matching is Pareto-efficient. However, students involved in interrupters are indifferent between applying or not to the school they interrupt with. Our contribution consists to make students strictly prefer one strategy to another to neutralize the interrupters.

3 Efficiency with cost

In this section, we study cost distributions that lead to an equilibrium such that the obtained matching is Pareto-efficient. Following our setting, a student i involved in an interrupter (i, s) is

[10] We discuss in the introduction several reasons why some students benefit from programs that waive application fees. Thus, we consider this possibility in our model.

[11] The most interesting case is when the student is rejected by the chain it has generated. In this case, a Pareto improvement is possible. However, we also address the issue in which the student is rejected because of a rejection chain that it did not generate.

[12] The term neutralize signifies that the student involved in an interrupter no longer applies to the school with which it interrupts.

indifferent between applying to school s or not, given the other students' strategy fixed. Suppose student i has a positive application cost to school s and i will never be assigned to s, then i's strategy will not include school s. Let us illustrate this by considering example 4 in Kesten (2010).

Example 1. Consider $\mathcal{I} := \{i_1, i_2, i_3\}$ and $\mathcal{S} := \{s_1, s_2, s_3\}$ and $\forall s \in \mathcal{S}, q_s = 1$. We denote by $(\cdot)$ that the school does not have a priority over the student set. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_1	$\cdot$
i_1	i_2	$\cdot$
i_2	i_3	$\cdot$

P_{i_1}	P_{i_2}	P_{i_3}
$\boxed{s_1}$	s_1	$\boxed{s_2}$
$\underline{s_2}$	s_2	$\underline{s_1}$
s_3	$\boxed{\underline{s_3}}$	s_3

The matching obtained using the true preferences is underlined in the students' preferences. In the Kesten approach, to reach efficiency, we must neutralize the interrupter (i_2, s_2). We construct a new strategy for student i_2 denoted by Q'_{i_2}, where Q'_{i_2} consists of applying to schools s_1 and s_3 only. By considering the strategy Q'_{i_2} instead of the original preferences P_{i_2}, we obtain the matching placed in a box.
With strategies Q_{i_1} and Q_{i_3} fixed, student i_2 is assigned to school s_3 using either the original strategy $Q_{i_2} = P_{i_2}$ or the new strategy Q'_{i_2}. We set a positive cost for student i_2 to apply to school s_2 ($c_{i_2,s_2} = 1$). Using the original strategy Q_{i_2}, student i_2 has to pay a positive cost to apply to school s_2, while using the new strategy Q'_{i_2}, the cost is zero. Since the payoff functions are lexicographic, student i_2 prefers the strategy Q'_{i_2} over the original strategy $Q_{i_2} = P_{i_2}$, and there are no profitable deviations.

Our first result is immediate with this example. We see that the strategy Q_{i_2} is not dominant. By considering $\mathcal{Q}_{-i_2}$ the strategy Q'_{i_2} is preferred to Q_{i_2} by i_2. We can generalize the result in the Proposition 1[13]:

Proposition 1. There exist cost profiles under which the truthful revelation strategy is not a dominant strategy.

Say differently: The truthful revelation strategy is not a strictly dominant strategy.
Therefore, studying students' strategic considerations when facing application cost implementation is necessary. A second fact emphasized in the Example 1 is that by using the approach suggested by Kesten (2010), cost implementation can lead to Pareto improvement. Throughout the Kesten algorithm, students are identified as involved in an interrupter at step t. To generalize the result, we introduce a new notation:

Definition 1. Let $\tilde{I}_t$ be the set of students involved in an interrupter and identified by the EADAM algorithm at step t. The EADAM algorithm identifies $\tilde{I}_t$ as follows:

(i) Run DA algorithm at step $t-1$ until the last step k in which a student involved in an interrupter is rejected from the school with which it interrupts.

[13] It is sufficient to consider Example 1 as proof of Proposition 1.

(ii) Define $\tilde{I}_t$ as the set of students involved in an interrupter and rejected from the schools with which they interrupt in step k.

We denote by s_i^t the school s with which student i interrupts at step t.

Let us illustrate our first Theorem by designing a cost profile and a strategy for each student for Pareto improvement at equilibrium.

- In the cost profile, we consider that costs are equal to zero for every application except those linked to an interrupter. Formally, $\forall i \in \mathcal{I}, \forall s \in \mathcal{S}$ such that $\exists t, i \in \tilde{I}_t$ and $s = s_i^t$ we have $c_{i,s} = 1$. $\forall i \in \mathcal{I}, \forall s \in \mathcal{S}$ such that $\nexists t, i \in \tilde{I}_t$ we have $c_{i,s} = 0$.
- Each student's strategy is as follows: for any $i \in \mathcal{I}$ such that $\nexists t, i \in \tilde{I}_t$, use the truthful strategy P_i. For any $i \in \mathcal{I}$ such that $\exists t, i \in \tilde{I}_t$, use the modified strategy Q'_i that drop s_i^t from P_i, thus $s_i^t \notin Q'_i$.

We thus have a situation in which no student can gain by a unilateral change of strategy if the strategies of the others remain unchanged. In this equilibrium, all interrupters are neutralized. Therefore the obtained matching is Pareto-efficient.

Theorem 1. There exists a cost profile C such that there exists a Nash equilibrium in undominated strategies that Pareto dominates the student optimal stable assignment.

Say differently: There exists a Nash equilibrium in undominated strategies that Pareto dominates the student optimal stable assignment.
A weakness of the cost profile presented is that students are not treated equally. Some students are charged an application fee at a given school, while others are not. However, no student pays an application cost at equilibrium using this profile and the outcome is Pareto-efficient. It is important to note that a student must only be identified once in the EADAM algorithm to face application costs. Our contribution lies in considering the students' strategies, which were not taken into account in the approach presented in Kesten (2010).

In the rest of the section, we determine which applications ensure equilibrium. To do this, we need to identify the applications that block deviations of students involved in interrupters. Intuitively, for student i involved in an interrupter (i, s) not to apply to school s, the rejection chain that results in i's rejection of s must still be possible. As mentioned, a student i involved in an interrupter generates a rejection chain. If the interrupter is neutralized, the rejection chain is not generated. Accordingly, all students in the chain have improved their situations. Indeed they are no longer rejected from the school to which they were tentatively assigned. It is, therefore, necessary to maintain the portion of the chain at the origin of the rejection of i to consider an equilibrium. Furthermore, when i is rejected from s by a rejection chain of which it was not the generator, then when the interrupter is neutralized, the students involved in the rejection chain that caused i to be rejected from s do not have a preferred assignment. Therefore, they may be assigned to the same school as in the previous step. This nuance leads to a change in students' strategies.
We have to determine which schools can be removed from students' strategies without harming them. Example 2 illustrates this statement.

Example 2. Let $\mathcal{I} := \{i_1, i_2, i_3, i_4\}$ and $\mathcal{S} := \{s_1, s_2, s_3, s_4\}$, $\forall s \in \mathcal{S}, q_s = 1$. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
i_3	i_4	i_1	$\cdot$
i_4	i_1	i_2	$\cdot$
i_1	i_2	i_3	$\cdot$
i_2	i_3	i_4	$\cdot$

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_1	s_1	s_2	s_3
s_2	s_2	s_3	s_1
s_3	s_3	s_1	s_2
s_4	s_4	s_4	s_4

Let us consider different possible strategies for i_2 using EADAM:

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
s_1^*	s_1	s_2^*	s_3^*
$\textcircled{$s_2$}$	s_2	$\textcircled{$s_3$}$	$\textcircled{$s_1$}$
$\underline{s_3}$	s_3	$\underline{s_1}$	$\underline{s_2}$
s_4	$\textcircled{$\underline{s_4}*	s_4	s_4

- At step 0 of the EADAM, the obtained matching is underlined in the strategy profile. The strategy of i_2 is given by: $Q_{i_2} : s_1, s_2, s_3, s_4, i_2$.
- The identified interrupter is (i_2, s_3). In step 1, the obtained matching is circled in the strategy profile. The strategy of i_2 is given by: $Q'_{i_2} : s_1, s_2, s_4, i_2$ and i_2 is assigned to s_4.
- The identified interrupter is (i_2, s_2). In step 2, the obtained matching is asterisked in the strategy profile. The strategy of i_2 is given by: $Q''_{i_2} : s_1, s_4, i_2$ and i_2 is assigned to s_4.

The outcome of the DA algorithm run at the end of step 2 is Pareto-efficient.
We can see that students i_1, i_3, and i_4 are indifferent between applying to s_4 or not. Thus, if the application cost is strictly positive for s_4, they prefer not to apply. In addition, there exists an equilibrium such that students apply to at most two schools. Using this strategy, and in the presence of a specific cost profile, the matching obtained is Pareto-efficient. To illustrate this, consider that the strategies of i_1 and i_3 are fixed $Q_{i_1} : s_1, s_2, s_3, i_1$ and $Q_{i_3} : s_2, s_3, s_1, i_3$. For student i_4, we study the strategies denoted Q_{i_4} and Q'_{i_4} such as $Q_{i_4} : s_3, s_1, s_2, i_4$ and $Q'_{i_4} : s_3, s_2, i_4$. For student i_2 we study the strategies $Q_{i_2} : s_1, s_2, s_4, i_2$ and $Q'_{i_2} : s_1, s_4, i_2$.

i_2 \ i_4	Q_{i_4}	Q'_{i_4}
Q_{i_2}	$\mu(i_2) = s_4; \mu(i_4) = s_1$	$\mu(i_2) = s_4; \mu(i_4) = s_2$
Q'_{i_2}	$\mu(i_2) = s_4; \mu(i_4) = s_3$	$\mu(i_2) = s_4; \mu(i_4) = s_3$

We can see that i_4 prefers the strategy Q_{i_4} to the strategy Q'_{i_4} when i_2 use the strategy Q_{i_2}. Because i_4 did not apply to the school where she was supposed to be assigned, namely s_1. In all situations, i_2 is assigned to s_4. Therefore, by adding a positive cost to i_2's application to s_2, the strategy will be Q'_{i_2}, as suggested in Theorem 1. Knowing this, i_4 is indifferent between Q_{i_4} and Q'_{i_4}. Therefore, in the presence of cost, i_4 will not apply to s_1. The fact that i_2 has no interest in applying to s_2 when i_4 does not apply to s_1 is because i_4 applies to s_2 and $i_4 \succ_{s_2} i_2$.

Consequently, we reduce the number of student applications and achieve efficiency. Using the same reasoning for the other students we have $\mathcal{Q}_{-i_2}$ with $Q_{i_1} : s_1, s_3, i_1$, $Q_{i_3} : s_2, s_1, i_3$ and Q_{i_4} :

s_3, s_2, i_4, student i_2 is indifferent between $Q_{i_2} : s_1, s_2, s_3, s_4, i_2, Q'_{i_2} : s_1, s_2, s_4, i_2, Q''_{i_2} : s_1, s_4, i_2$ and $Q^{\Delta}_{i_2} : s_4, i_2$. Given that these strategies have the same matching outcome to i_2, the chosen strategy will be $Q^{\Delta}_{i_2}$ in the presence of costs. The matching is Pareto-efficient.

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
s_1	s_1	s_2	s_3
	s_2		
$\underline{s_3}$	s_3	$\underline{s_1}$	$\underline{s_2}$
	$\underline{s_4}$		

Q_{i_1}	$Q^{\Delta}_{i_2}$	Q_{i_3}	Q_{i_4}
$\underline{s_1}$		$\underline{s_2}$	$\underline{s_3}$
s_3		s_1	s_2
	$\underline{s_4}$		

Proposition 2 generalizes this reasoning. In the remainder of this paper, we denote by $\mu_{PE}(i)$ the school to which student i is assigned in the Pareto-efficient matching. We use the notation $\mu_I(i)$ in the matching obtained using true preferences. Concerning the students' strategies, let us consider the following profile: for every $i \in \mathcal{I}$ such that $\mu_{PE}(i) \neq \mu_I(i)$, we have $Q_i : \mu_{PE}(i), \mu_I(i), i$. For every $i \in \mathcal{I}$ such that $\mu_{PE}(i) = \mu_I(i)$, we have $Q_i : \mu_I(i), i$. This means that all students use a strategy where their reported preferences are composed of the school they are assigned in the Pareto-efficient outcome and the school they are assigned using DA. If the two schools are the same, students only report one school in their preferences.
We can set an application cost profile such that $\forall i \in \mathcal{I}, \forall s \in \mathcal{S}$ such that $\mu_I(i) = s, c_{i,s} = 0$. And $\forall i \in \mathcal{I}, \forall t \in \mathcal{S}$ such that $\mu_I(i) \neq t, c_{i,t} = 1$. Knowing that application costs are very low, we consider that it is profitable to apply to a school where the students prefer $\mu_I(i)$ if they are guaranteed to be assigned. This extreme case leads us to a new result:

Proposition 2. For every strategy profile that respects:

(i) for every step t in the EADAM algorithm, for every student $i \in \tilde{I}_t$ we have $s_i^t \notin Q_i$ and,

(ii) for every student $i \in \mathcal{I}$ we have $\mu_I(i), \mu_{PE}(i) \in Q_i$, ranked according to true preferences,

there exists a Nash equilibrium that is not dominated, and Pareto dominates the student optimal stable assignment.

Condition (i) follows from Theorem 1. Proposition 2 has substantial implications. For instance, if there is an equilibrium such that the outcome is Pareto-efficient, then it is possible to identify another equilibrium with at most two schools reported in the students' preferences using DA, as presented in Example 2. This is true for any set of students and schools. It has consequences on the number of applications to consider for schools. For this, it is required to use a profile of application fees that considers individuals' preferences.
For condition (ii), we establish that it is sufficient to include the assigned school using DA with the true preferences in the modified preferences to block profitable deviations of students involved in an interrupter (Lemma 2[14] in Appendix). Thus, a new result can be formulated. As a consequence of the Proposition 2 and using the construction of the EADAM, we obtain that all the steps of the Kesten algorithm are Nash implementable, and Pareto dominates the previous steps. In the EADAM algorithm, the interrupters are identified successively, and the order has importance. More precisely, if we consider an interrupter (i, s) identified at step k, if

[14] For this, all students must apply to the school to which they are assigned using DA with true preferences. In addition, the preference order must be respected.

the student i deviates at step $k+n$ with $n \geq 1$ by applying to s, the obtained assignment is the one at step k. Thus, considering the neutralization of a subset of interrupters always leads to an equilibrium, but the outcome of this equilibrium is determined by the neutralized interrupters. This result is expressed in the following proposition.

Proposition 3. Suppose $\tilde{I}'_t \subseteq \bigcup_{t\in\{1\ldots n\}} \tilde{I}_t$ is a subset of interrupter. It exists k such that for every $h \leq k$, $(i, s_i^h) \in \tilde{I}'_t$ and $(i, s_i^{k+1}) \notin \tilde{I}'_t$ then if each interrupter in $\tilde{I}'_t$ is neutralized and the preferences of other students remain unchanged, the obtained assignment is the one given by the EADAM at step k and it is an equilibrium using DA.

However, if costs are distributed in such a way that they are strictly positive for every application, this does not guarantee that $\mu_I(i)$ is in the strategy Q_i. We consider this case in Section 5.

4 The Boston mechanism and the Top Trading Cycles mechanism with cost

We observe that in DA the costs impact the Nash equilibria. We have for the moment presented the Pareto improvements that are possible through costs implementation. We investigate if similar effects can be expected in other mechanisms often used in the literature. In this section, we study the implementation of application costs in the Boston mechanism and the TTC mechanism.

4.1 The Boston mechanism

The Boston mechanism[15] has several nice properties: it is individually rational, non-wasteful, and Pareto-efficient. However, it is neither stable nor strategy-proof. Moreover, Ergin and Sönmez (2006) have shown that the preference revelation game induced by DA has a dominant strategy equilibrium (truthful-revelation), and its outcome is either equal to or Pareto dominates the Nash equilibrium outcomes of the Boston mechanism. Say differently the DA outcome is the best one that students can hope for under the Boston mechanism.
We now aim to identify a profile of low application fees that would enable to reach equilibrium with a Pareto-efficient outcome. We know that the matching, using true preferences, suggested by the Boston mechanism is Pareto-efficient but it is not necessarily an equilibrium.

A result by Haeringer and Klijn (2009) suggests that the set of Nash equilibrium outcomes does not depend on the number of schools in the students' preferences. All Nash equilibrium outcomes are possible regardless of the number of schools reported by students. Thus, we can consider the case where each student reports only one school.
In DA, we notice that the students' strategy is to block the deviations of the students involved in the interrupters. However, this is not possible in the Boston mechanism. We consider application costs to be low. By our definition, these costs do not alter the ranking of schools in students' preferences. Therefore, since we preserve the set of Nash equilibrium outcomes with only one

[15]The term "Boston mechanism" was introduced by Abdulkadiroğlu and Sönmez (2003) because the mechanism was used in the Boston school district until recently.

school in the students' strategy, it is intuitive that the costs do not allow for an improvement in the situation. Implementing low costs does not create an equilibrium.
Let $\mathcal{Q}^\beta$ denote an equilibrium using the Boston mechanism.

Proposition 4. For every cost profile C under which it exists an equilibrium $\mathcal{Q}^\beta$ (in complete information) such that μ is the equilibrium outcome and it is Pareto-efficient, if and only if $\mathcal{Q}^\beta$ is an equilibrium using the Boston mechanism and the outcome is Pareto-efficient in the absence of cost.

The consequence of Proposition 4 is that setting low costs for application is not necessary when using the Boston mechanism. Indeed, it is impossible to guarantee the creation of a new equilibrium with a low cost profile. If an equilibrium leads to a Pareto-efficient outcome, then it is not through low costs implementation. By using another definition of the payoff function, this would be possible. It would then be necessary to implement costs to change the order of schools in the student's preferences. A follow-up research would be to set high application costs for every school except the one to which the student is assigned in the Pareto-efficient case. However, this is beyond the scope of this paper.

4.2 The Gale's Top Trading Cycles mechanism

We know that TTC[16] is Pareto-efficient, strategy-proof, individually rational, and non-wasteful. However, it is not stable, which is an essential property for matching sustainability. It is known from the literature that stability and Pareto-efficiency are incompatible (Roth, 1982).
Similar to the Boston mechanism, the set of outcomes of the Nash equilibria does not depend on the number of schools reported in the students' preferences. This result is determined by Haeringer and Klijn (2009). All equilibrium outcomes are possible with only one school in the students' preferences. Thus there is an equilibrium that produces a Pareto-efficient outcome. Adding costs would result in a reduction in students' payoff but would not improve their situation. It is intuitive to see that the number of schools that constitute the students' strategy would be one and that the obtained matching is identical to the one using the true preferences.
Let $\mathcal{Q}^\tau$ denote an equilibrium using the TTC mechanism.

Proposition 5. For every cost profile C under which it exists an equilibrium $\mathcal{Q}^\tau$ (in complete information) such that μ is the equilibrium outcome and it is Pareto-efficient, if and only if $\mathcal{Q}^\tau$ is an equilibrium using the TTC mechanism and the outcome is Pareto-efficient in the absence of cost.

Proposition 5 implies that implementing application costs will never positively affect students using the TTC mechanism. Considering higher costs, changing the students' preferences would be possible. Therefore, the matching at equilibrium would be stable and Pareto-efficient.

[16]The TTC mechanism was first introduced in the context of school choice in Abdulkadiroğlu and Sönmez (2003).

5 Cost profile conditions

In this section, we investigate the condition such that the cost profile leads to a Pareto-efficient outcome at equilibrium using DA. In Example 2, we permitted costs to be positive for almost all applications. The condition set is that the costs must be zero for the schools assigned using the true preferences and that the student's situation can be improved. Formally $\forall i \in \mathcal{I}, \forall s \in \mathcal{S}$ such that $\mu_I(i) = s$ and $s \neq \mu_{PE}(i)$ we have $c_{i,s} = 0$. However, this condition is mandatory to maintain a Pareto-efficient equilibrium outcome.

Example 3. Let us consider Example 2 with the following strategy profile:

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
$\underline{s_1}$		$\underline{s_2}$	$\underline{s_3}$
s_3		s_1	s_2
	$\underline{s_4}$		

Considering strictly positive application fees, we know that the strategy used by i_2 is $Q_{i_2} : s_4, i_2$. Let us set a positive cost for i_1's application to school s_3, $c_{i_1,s_3} = 1$. Knowing the strategy Q_{i_2}, we can consider two strategies for student i_1 producing the same assignment: $Q_{i_1} : s_1, s_3, i_1$ and $Q'_{i_1} : s_1, i_1$. In the case of the positive cost, i_1 will prefer the strategy Q'_{i_1}.

Q'_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
$\underline{s_1}$		$\underline{s_2}$	$\underline{s_3}$
		s_1	s_2
	$\underline{s_4}$		

These strategies are not an equilibrium, for example, i_2 has an interest in changing its strategy by applying to s_2.

Q'_{i_1}	Q'_{i_2}	Q_{i_3}	Q_{i_4}
s_1		s_2	$\underline{s_3}$
	$\underline{s_2}$		
		$\underline{s_1}$	s_2
$\underline{i_1}$	i_2	i_3	i_4

Therefore, students must apply to the school they have been assigned with DA using true preferences. This is true even for schools that are not involved in an interrupter.

Theorem 2 states the result illustrated in Example 3. Let $\mathcal{Q}_T$ denote the equilibrium such that $\forall i \in \mathcal{I}, Q_i = P_i$ using μ in the problem $(P, \succ, q)$. Let $\mathcal{Q}_{PE}$ denote an equilibrium that provides a Pareto-efficient outcome using DA in the problem $(\mathcal{Q}_{PE}, \succ, q)$. We denote $\mu_I(i)$ and $\mu_{PE}(i)$ the respective outcome for every $i \in \mathcal{I}$.
This implication identified in Example 3 can be generalized. More precisely, here is a new theorem.

Theorem 2. If there exists a student $i \in \mathcal{I}$, such that $\mu_I(i) \neq \mu_{PE}(i)$ and $c_{i,\mu(i)} = 1$ then, there is no equilibrium $\mathcal{Q}$ that Pareto-dominates the DA outcome with the cost profile.

Theorem 2 must be interpreted as follows: if there exists a student such that the school to which she is assigned in the Pareto-efficient outcome is different from the stable outcome and that it is costly for her to apply to the school obtained in the stable outcome, then there is no Pareto-efficient equilibrium in the set of possible equilibria. Say differently, in the absence of costs, if there exists a student $i \in \mathcal{I}$, such that $\mu_I(i) \neq \mu_{PE}(i)$ and $\mu_I(i) \notin Q_i$ then, there is no equilibrium $\mathcal{Q}$ that Pareto-dominates the student optimal stable assignment.

Theorem 2 has substantial implications.

Corollary 1. If there exists a school $s \in \mathcal{S}$, such that $\mu_I^{-1}(s) \neq \mu_{PE}^{-1}(s)$ and for every $i \in \mathcal{I}, c_{i,s} = 1$ then, there is no equilibrium $\mathcal{Q}$ that Pareto-dominate the DA outcome with the cost profile.

Unlike Theorem 2, in Corollary 1, the application cost is positive for every student at school s and, therefore, for students being assigned to s using true preferences. It means that if different students are assigned to a school when the assignment is obtained using the true preferences of when it is Pareto-efficient, then if that school implements positive application costs for every student, the equilibrium resulting in a Pareto-efficient outcome is no longer an equilibrium. This is a major insight. Stated differently, this means that if application costs are set with respect to schools and not with respect to students and schools, there are only two cases in which a Pareto-efficient matching can be reached at equilibrium. The first one is if the matching obtained with the true preferences was Pareto-efficient. And the second is in the case the students set assigned to the school in both equilibriums are identical. Students are, therefore, negatively impacted by the costs because the matching does not change, but students now have to pay an application fee. Hence, setting a positive application cost for a school always has a negative impact on students. To illustrate Theorem 2, we can use the previously mentioned strategy. Each student apply to two schools $\mu_I(i)$ and $\mu_{PE}(i)$ and is assigned to $\mu_{PE}(i)$. If $\mu_I(i) = \mu_{PE}(i)$ they apply only to $\mu_I(i)$ and are assigned to it. The application to $\mu_I(i)$ is to block the possible deviations of the students involved in interrupters. This has a major consequence in the absence of cost, and we can reformulate the result as follows:
For every equilibrium such that every student i applies only to both $\mu_{PE}(i)$ and $\mu_I(i)$ and ranks schools between the two (or not), using the correct ranking, all the outcomes lead to the same Pareto-efficient assignment.
Another consequence of Theorem 2 is provided if we consider that all schools have positive application costs. We obtain that there is no equilibrium producing a Pareto-efficient outcome unless the outcome obtained using true student preferences is Pareto-efficient.

Corollary 2. If for every $i \in \mathcal{I}$ and for every $s \in \mathcal{S}, c_{i,s} = 1$ it exists an equilibrium $\mathcal{Q}_{PE}$ with an outcome μ_{PE} that is Pareto-efficient, if and only if for every $i \in \mathcal{I}, \mu_I(i) = \mu_{PE}(i)$ meaning that the truthful revelation strategy is an equilibrium with a Pareto-efficient outcome in the absence of costs.

Specifically, for schools such that $\mu_I^{-1}(s) = \mu_{PE}^{-1}(s)$ we can implement $\forall i \in \mathcal{I}, c_{i,s} = 1$ for every student. This is mentioned in Corollary 1. In other words, schools such that assignments are not modified when we consider the Pareto-efficient matching can set a positive application cost. However, this would not improve students' utility. Only the students in $\mu_I^{-1}(s)$ would be negatively impacted by these costs. Other students will not apply to the school s. Thus we find that setting positive application costs for every student at a given school is never beneficial for students.

Example 4. From Example 3, we observe that under specific conditions, by considering profitable deviations, no equilibrium is possible because of the cost and strategy profile used. Let us consider a cost profile such that $c_{i_2,s_2} = 1, c_{i_2,s_3} = 1$ and $c_{i_1,s_3} = 1$. Using the same strategies for the students, we obtain that the strategies used by i_1 and i_2 will rotate between the two following strategy profiles:

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
$\underline{s_1}$		$\underline{s_2}$	$\underline{s_3}$
s_3		s_1	s_2
	$\underline{s_4}$		

Q'_{i_1}	Q'_{i_2}	Q_{i_3}	Q_{i_4}
s_1		s_2	$\underline{s_3}$
	$\underline{s_2}$		
		$\underline{s_1}$	s_2
$\underline{i_1}$	i_2	i_3	i_4

Knowing that i_2 does not apply to s_2 and s_3, i_1 is indifferent between applying to s_3 or not. The application cost being positive, i_1 will prefer not to apply. Knowing that i_1 does not apply to s_3, the student i_2 has an interest in applying to s_2, which leads i_1 to be unassigned. i_1 now has a profitable deviation which is to apply to s_3. Knowing that i_1 applies to s_3, i_2 has no interest in applying to s_2 because of the positive cost. By repeating this reasoning, it is not possible to reach an equilibrium. There are, however, possible equilibria with this cost profile. For instance:

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
$\underline{s_3}$	$\underline{s_4}$	$\underline{s_1}$	$\underline{s_2}$

There are no profitable deviations according to this strategy profile and the cost profile, then it is an equilibrium. The outcome of this equilibrium is equal to the DA outcome without costs.

The following theorem states this result in general settings.

Theorem 3. For every cost profile C, there is at least one equilibrium with an outcome that all students weakly prefer to DA.

The difference between Theorem 2 and Theorem 3 is the constraint set on the cost profile. It is sufficient that a student i, assigned to two different schools at the equilibrium $\mathcal{Q}_T$ and $\mathcal{Q}_{PE}$, has a positive application cost at $\mu_I(i)$ to prevent the existence of a Pareto-efficient equilibrium. The set of possible Nash equilibria is impacted. By adding the constraint that all students involved in interrupters have a positive application cost at the school they interrupt with, as presented in Section 3, the set of possible Nash equilibria is further restricted. A major implication is therefore that the application cost profile, nonetheless low, impacts the set of possible equilibria. However, there is always an equilibrium such that students are assigned to the same schools as they would be using DA without costs. This equilibrium is used if at least one student has a positive application cost for the school assigned with the true preferences and she is assigned to another school in the Pareto-efficient outcome. In the equilibrium suggested in Example 4, the student concerned has to pay a positive cost (here i_1) and is assigned to the same school as in DA without cost. There is, therefore, a negative impact.

6 Discussion

As mentioned in the introduction, different organizations may waive application fees for selected students. We identify two main actors: schools and public administration. It is easy to see that the goals of these two actors are sometimes different. While a public policymaker would aim for market efficiency, schools would want the best possible students to be assigned to it. We have considered that only students can act strategically. A different analysis would be to consider school preferences rather than considering priorities; schools become strategic, and we can consider welfare criteria. Therefore, the school's objective is to maximize its welfare. We consider for every school s that s want to obtain the best possible students set $(\mu_I^{-1}(s))$. It is known from the literature that in the case of student proposing, schools can misreport their preferences to achieve a better match. We then consider that the schools' preferences are an equilibrium and that the schools' strategy now consists only in setting positive or zero application costs. Using true student preferences, schools are assigned the best student they can expect in a stable assignment. Considering a Pareto improvement, by neutralizing the interrupters, schools are therefore assigned to students who are less preferred by them. A possible interpretation for the rejection chains is that students get worse off when a chain is active, while schools get better off. We do not consider that application costs are transferred to schools. Otherwise, the school would always try to maximize its payoff and thus set positive fees. Furthermore, we do not consider that it is costly for schools to review applications. Thus, we focus on the schools' interest regarding the students assigned to it. We identify four possible strategies:

(i) Set a zero cost for every application.

(ii) Set a positive cost for student applications except for the best students the school can expect (i.e., students assigned to the school with true preferences do not pay an application fee).

(iii) Set a positive cost for applications from students who are the best students the school can expect and a zero cost for others (i.e., only students assigned to the school with true preferences pay an application fee).

(iv) Set a positive cost for every application.

If it is not costly for a school to review applications, then the school is indifferent between strategies (i) and (ii) and between strategies (iii) and (iv). If it is costly for a school to consider applications, then strategy (i) will never be used. An intuitive approach would be for each school to set a positive application cost for every student except the best students they can expect, as presented in strategy (ii). This could then make the school more attractive to the student concerned. But by doing this, the interrupters are neutralized, and the condition of zero costs for the students assigned with the true preferences presented in Theorem 2 is satisfied. The matching Pareto-dominates DA from the student's perspective, implying that the schools are negatively impacted. Students less preferred by the schools are assigned to them than in the case of DA without costs. The reasoning is identical for strategy (i). The condition cannot be met in the remaining two strategies because students assigned to the school using true preferences pay a positive cost. By using rejection chains, schools improve their situations. Thus the interest of schools is to maintain the interrupters. The strategy of the schools must therefore be to set a zero cost for the students involved in an interrupter with it. We might even consider a strategy where schools subsidize the students with whom they interrupt. This would be modeled as a negative cost. In this case, the student would apply to the school even if she is not accepted. Students would be assigned to the same school as using the true preferences. However, students would be charged a positive application fee. It is important to note that the school can prevent the condition from being met by itself. If the assignment is the same in the Pareto-efficient outcome and in the no-cost outcome, the school is indifferent between setting a cost or not. If the two outcomes are different for a school, it only needs to set a positive cost for the students in $\mu^{-1}(s)$ to prevent equilibria that would Pareto dominate DA without costs.

Proposition 6. Using strategies (iii) and (iv), schools always obtain an assignment that is weakly preferred to DA student-proposing, at equilibrium, for any other schools' strategies.

We identified a Pareto-efficient equilibrium where students apply to a maximum of two schools. This reduces the number of applications to be considered. Thus, a decision maker could design a cost profile as presented in Proposition 2. As a result, the students' situation improves, and the schools review a limited number of applications. It is essential for the decision-maker to centralize the processing of application fees. If the interrupters are neutralized and the condition presented in Theorem 2 is not satisfied, no Pareto improvement is possible. Formulated differently, if the public administration aims at achieving Pareto-efficient matching, it can be blocked by schools. Thus, schools have a significant strategic impact on possible matching.

7 Concluding Remarks

In this paper, we studied the impact of application costs on students' strategies. We first show that application costs impact the dominant strategy, namely truthful revelation, using DA. At the same time, we introduced a cost profile and a strategy leading to a Pareto-efficient outcome at equilibrium, that Pareto dominates the DA outcome. We have identified two conditions on the application cost profile that permit a Pareto improvement by considering strategic deviations. We also presented an equilibrium with a Pareto-efficient outcome where each student applies to at most two schools. This, therefore, improves the situation for students and schools with fewer applications to review. By analyzing other mechanisms often mentioned in the literature, we have shown that application costs do not impact the Boston and TTC mechanisms. Thus, when these mechanisms are employed, application costs do not alter matchings but reduce student welfare because they face fees. Implementing an application fee waiver to improve market efficiency is not relevant for these mechanisms, contrary to DA. We then studied the conditions that costs must satisfy to have a Pareto-improvement. One main consequence is that students do not face the same costs. Specifically for a given school, some students must have the fees waived and others not. This makes implementing such a policy difficult, as complete information is needed.

At the same time, since schools generally set application costs themselves, we have shown that this allows them to be strategic. They can independently block Pareto improvements and have students they prefer assigned to them. This paper, therefore, highlights the importance of application cost waiver management.

Further research may consider higher application costs, which relaxes our assumption. In this case, the impact on strategies would be different, and mechanisms like Boston or TTC could be impacted. Considering an incomplete information approach would be beneficial to the literature. Ehlers and Massó (2015) proposes a method for switching from a complete to an incomplete information framework under specific conditions. These conditions are mainly based on stability, which is incompatible with our approach. Addressing the impact of application costs on students in an incomplete information framework would provide fundamental insights for policy implementation.

Appendix

Proof of Theorem 1

Proof. We denote by P_i the true preference list and Q_i any generic preference list for every student i. For any $\mathcal{Q}$, let $K_i^{\mathcal{Q}}$ be the list Q_i where the schools s such that (i, s) is an interrupter have been dropped. We know that under DA we have:

(i) DA is strategy-proof by Gale and Shapley (1962): $DA_i(P_i, \mathcal{Q}_{-i}) R_i DA_i(Q_i, \mathcal{Q}_{-i}), \forall Q_i, \forall \mathcal{Q}_{-i}$,

(ii) By Proposition 3 by Kesten (2010): $DA_i(K_i^{\mathcal{Q}}, \mathcal{Q}_{-i}) = DA_i(Q_i, \mathcal{Q}_{-i}), \forall Q_i, \forall \mathcal{Q}_{-i}$,

(iii) By Theorem 1 by Kesten (2010): $DA((K_i^P)_{i=1,\dots,n})$ Pareto dominates $DA((P_i)_i)$.

We deduce that $DA_i(K_i^P, K_{-i}^P) = DA_i(P_i, K_{-i}^P) R_i DA_i(Q_i, K_{-i}^P), \forall Q_i$, by (ii) and (i).

Hence $(K_i^P)_{i=1,\dots,n}$ is an equilibrium, and Pareto dominates the DA outcome under P form (iii). □

Lemma 1. Using the EADAM algorithm if $\mu PE^{-1}(s) \neq \emptyset$, then $\mu_I^{-1}(s) \neq \emptyset$. In addition for every $s \in \mathcal{S}$ where s is equal to s_i^t for some step t in EADAM and some $i \in \tilde{I}_t$ we have $\mu_{PE}^{-1}(s^*) \neq \emptyset$.

Proof. By contradiction, suppose that $\mu_I^{-1}(s) = \emptyset$ and $\mu_{PE}^{-1}(s) \neq \emptyset$, we have $\mu_I^{-1}(s) \neq \mu_{PE}^{-1}(s)$. This implies that the outcome obtained with equilibrium $\mathcal{Q}$ is not stable, as there exists a student i who prefers s to their assigned school $\mu_I(i)$ under $\mathcal{Q}PE$ but not under $\mathcal{Q}$. This violates the non-wastefulness property of the equilibrium. Then i has a profitable deviation due to which i is assigned to s. Thus $\mathcal{Q}$ is not an equilibrium, which is a contradiction. Then we know that if $\mu_{PE}^{-1}(s) \neq \emptyset$ then $\mu_I^{-1}(s) \neq \emptyset$.

Consider that there exists an interrupter (j, s^*). Suppose that $\mu_I(i) = s^*$ and assume for contradiction that no student under EADAM is assigned to s^*, meaning that there does not exist any $i' \in \mathcal{I}$ such that $\mu_{PE}(i') = \mu_I(i)$. This would prevent the rejection of j from school s^*. Suppose no students are assigned to s^* under EADAM. In that case, it means that neutralized (j, s^*) did not improve the situation of the students present in the rejection chain generated by j by applying to s^*, so s^* is not assigned to i' under the EADAM which leads to a contradiction. □

Lemma 2. Using the truthful preference order, if all students include $\mu_I(i)$ in their application strategies, there exists a rejection chain that blocks the deviations of students involved in an interrupter.

Proof. Consider a student i who is assigned to s at the end of the EADAM algorithm, ($\mu_{PE}(i) = s$), and that this school was involved in an interrupter pair with student j. So we know that $j \succ_s i$ and $sP_j\mu_{PE}(j)$. (If the school was not involved in an interrupter pair, then there is no student i' such that $sP_{i'}\mu_{PE}(i')$ and $i' \succ_s i$. The proof would then be immediate.). Consider that j changes its strategy and applies to s. Thus j leads to the rejection of i from school s. Then i applying to school s' such that $\mu_I(i) = s'$ results in the rejection of a student i' who was assigned to s' using the EADAM algorithm, such that $\mu_{PE}(i') = s$. Then i' rejects i'' from s'' such that $\mu_{PE}(i'') = s''$... until j is rejected from s. By contradiction, suppose that i does not reject anyone or does not generate a rejection chain that allows j to be rejected from s.
Let us temporarily consider the DA process with the true preferences. We know that j is an interrupter, so there exists a student; let us note k such that k is applying to school s and causes j to be rejected at a later step. And k is assigned to s, such that $\mu_I(k) = s$, otherwise k is also an interrupter (and k is the last interrupter to be rejected, and consequently EADAM identifies k as the interrupter to be neutralized). If k is not in the rejection chain generated by j, then k will apply to s at the same step as when it is rejected j. Suppose that (j, s) is neutralized, then i is rejected at a later step by k and $\mu_I(k) = s$. Because we have $k \succ_s j$ and $j \succ_s i$ which implies $k \succ_s i$. Therefore, i can't be assigned to s if i does not generate the rejection chain that rejects j from s. This means that k must have been rejected of a school by j's application to school s and rejected j.
Consequently, we know that there exists a school, let us note s^*, to which i applied such that sP_is^* and $s^*R_i\mu_I(i)$ that generated a rejection chain which rejected j from s. Let us consider two cases:

- $s^* \neq \mu_I(i)$, then i was an interrupter by applying to s^* because i is not assigned to s^* and has rejected a student from s^* (who then lead to the rejection of j from s). We know from Lemma 1 that i is indifferent between applying to s^* or not.
- $s^* = \mu_I(i)$, then i can generate the same rejection chain if i includes $\mu_I(i)$ in its application strategy.

So we know that i generates its rejection chain (which will lead to the rejection of j from s) by applying to $\mu_I(i)$. □

Proof Proposition 2

Proof. Let us assume that each student plays the strategy mentioned. As a reminder: $\forall i \in \mathcal{I}$ such that $\mu_{PE}(i) \neq \mu_I(i)$, Q_i with $Q_i : \mu_{PE}(i), \mu_I(i), i$. $\forall i \in \mathcal{I}$ such that $\mu_{PE}(i) = \mu_I(i)$, Q_i with $Q_i : \mu_I(i), i$.
The objective of the proof is to show that the deviation from the strategy is not beneficial for a student involved in an interrupter. (For the other students, we know that the obtained matching is stable, so there is no profitable deviation). Here is the reasoning used for this. The schools obtained under DA must be assigned under EADAM (Lemma 1). Then, it is necessary

to guarantee the existence of the rejection chain that leads the student to be rejected in case of deviation (Lemma 2).

From Lemma 1, we know that there exists a student i' such that $\mu_I(i) = \mu_{PE}(i')$. i' by being rejected from $\mu_I(i)$ will accordingly apply to $\mu_I(i')$. Using Lemma 3, we have the same reasoning; the rejection chain will continue until j is rejected from s. Considering a strategy where each student applies the school to which she is assigned under DA is therefore sufficient to block j from being assigned to s.

Thus, j is indifferent between applying to school s or not when other students apply to only two schools (the ones to which they are assigned under DA and in the Pareto-efficient case). In the presence of cost, j will not apply to s. Therefore, the matching will be Pareto-efficient. Moreover, according to the cost profile, it is an equilibrium. □

Proof Proposition 3

Proof. Let us assume that each student plays the strategy mentioned. As a reminder: $\forall i \in \mathcal{I}$ such that $\mu_{PE}(i) \neq \mu_I(i)$, Q_i with $Q_i : \mu_{PE}(i), \mu_I(i)$. $\forall i \in \mathcal{I}$ such that $\mu_{PE}(i) = \mu_I(i)$, Q_i with $Q_i : \mu_I(i)$.

The objective of the proof is to show that the deviation from the strategy is not beneficial for a student involved in an interrupter. (For the other students, we know that the obtained matching is stable, so there is no profitable deviation). Here is the reasoning used for this. The schools obtained under DA must be assigned under EADAM (Lemma 1). Then, it is necessary to guarantee the existence of the rejection chain that leads the student to be rejected in case of deviation (Lemma 2).

From Lemma 1, we know that there exists a student i' such that $\mu_I(i) = \mu_{PE}(i')$. i' by being rejected from $\mu_I(i)$ will accordingly apply to $\mu_I(i')$. Using Lemma 3, we have the same reasoning; the rejection chain will continue until j is rejected from s. Considering a strategy where each student applies the school to which she is assigned under DA is therefore sufficient to block j from being assigned to s.

Thus, j is indifferent between applying to school s or not when other students apply to only two schools (the ones to which they are assigned under DA and in the Pareto-efficient case). In the presence of cost, j will not apply to s. Therefore, the matching will be Pareto-efficient. Moreover, according to the cost profile, it is an equilibrium. □

Proof of Proposition 4

Proof. In this proof, let us note $\mu_\beta(i)$ for every student i the matching obtained with an equilibrium $\mathcal{Q}$ and $\mu_{\beta PE}(i)$ the matching obtained which is Pareto-efficient, using the Boston mechanism. Pareto-efficient matching is obtained using the students' true preferences. We consider the following cases in absence of costs. Note that Proposition 5.2 of Haeringer and Klijn (2009) allows us to consider strategies for each student with a single school.

Case 1: The matching obtained at equilibrium provides a Pareto-efficient outcome. Then $\forall i \in \mathcal{I}, \mu_\beta(i) = \mu_{\beta PE}(i)$. Then there is a cost profile that conserves this equilibrium, such that, for instance: $\forall i \in \mathcal{I}, \forall s \in \mathcal{S}, c_{i,s} = 0$.

Case 2: The equilibrium outcome is not Pareto-efficient. Then $\exists i \in \mathcal{I}$ such that $\mu_\beta(i) \neq \mu_{\beta PE}(i)$. From (Theorem 1. Ergin and Sönmez (2006)) we know that the set of Nash equilibrium outcomes is equal to the set of stable matching under the true preferences. Then we know that $\forall i \in \mathcal{I}, \forall s$ such that $sP_i\mu_\beta(i), \exists i' \in \mathcal{I}$ such that $i' \succ_s i$. By contradiction, let us suppose that there exists a cost profile, C that allows an equilibrium $\mathcal{Q}^*$ with $\mu^*_\beta(i)$ the matching obtain by using the equilibrium $\mathcal{Q}^*$ such that $\forall i \in \mathcal{I}, \mu^*_\beta(i)R_i\mu_\beta(i)$ and $\exists i \in \mathcal{I}, \mu^*_\beta(i)P_i\mu_\beta(i)$. We denote by $Q_i : \mu_\beta(i), i$ and $Q^*_i : \mu^*_\beta(i), i$ the i strategies under the equilibrium $\mathcal{Q}$ and $\mathcal{Q}^*$ respectively. Let us consider a student i such that $\mu^*_\beta(i) \neq \mu_\beta(i)$. By definition we have $\mu^*_\beta(i)P_i\mu_\beta(i)$. By stability we know that $\exists j \in \mathcal{I}, \mu^*_\beta(i) = \mu_\beta(j)$. Then, at $\mathcal{Q}^*$ we have $\mu^*_\beta(j)R_j\mu_\beta(j)$. On the other hand, the obtained matching is no longer stable and there is a student who can be assigned to the school by ordering the school to the first position in its strategy. Therefore, either $\mathcal{Q}$ or $\mathcal{Q}^*$ is not an equilibrium, which leads to a contradiction.
This proof uses mainly the theorem of Ergin and Sönmez (2006). In conclusion, this proof demonstrates that costs do not generally lead to a Pareto-efficient equilibrium outcome unless the equilibrium is already Pareto-efficient in the absence of costs. □

Proof of Proposition 5

Proof. From Haeringer and Klijn (2009), we know that all equilibrium outcomes are possible with only one school in the students' strategy using the TTC. Then we consider that $\forall i \in \mathcal{I}, Q_i : \mu_\tau(i), i$, with $\mu_\tau(i)$ the school to which i is assigned at equilibrium $\mathcal{Q}$.
Case 1: The matching obtained at equilibrium is Pareto-efficient. Then there is a costs profile that conserves this equilibrium, for instance: $\forall i \in \mathcal{I}, \forall s \in \mathcal{S}, c_{i,s} = 0$.
Case 2: The equilibrium outcome is not Pareto-efficient. By contradiction, let us suppose that a cost profile exists, C, such that $\mu_\tau(i)$ is Pareto-efficient. We know that the costs are low, so the order of schools in students' preferences cannot be changed due to the implementation of costs. Then $\mathcal{Q}$ is also equilibrium, and the outcome is Pareto-efficient in the absence of cost, and this is a contradiction. □

Proof Theorem 2

Proof. We denote by μ the outcome of the equilibrium $\mathcal{Q}_T$ and μ_{PE} the outcome of the equilibrium $\mathcal{Q}_{PE}$, which is Pareto-efficient. Since $\mu_I^{-1}(s) \neq \mu_{PE}^{-1}(s)$ we know that $\exists i \in \mu_I^{-1}(s)$ such that $\mu_I(i) \neq \mu_{PE}(i)$ with $\mu_I(i) = s$ and $\mu_{PE}(i)P_is$ as preferences are strict. Using equilibrium $\mathcal{Q}_{PE}, i$ is assigned to $\mu_{PE}(i)$. We denote by Q_{PE_i} the strategy of i under the equilibrium $\mathcal{Q}_{PE}$. If we set a positive cost such that $c_{i,s} = 1$ we know that considering $\mathcal{Q}_{PE-i}$, i prefer the strategy Q'_{PE_i} to the strategy Q_{PE_i} with $s \notin Q'_{PE_i}, \mu_{PE}(i) \in Q'_{PE_i}$, and $s, \mu_{PE}(i) \in Q_{PE_i}$. As the outcome of $\mathcal{Q}_T$ is stable and the outcome of $\mathcal{Q}_{PE-i}$ is unstable we know that $\exists i^*, j \in \mathcal{I}$ such that $j \succ_{\mu_{PE}(i^*)} i^*$ and $\mu_{PE}(i^*)P_j\mu_{PE}(j)$ (in fact j is a student involved in the interrupter $(j, \mu_{PE}(i^*))$). Considering the strategy profile $(Q'_{PE_i}, \mathcal{Q}_{PE-i})$, it is no longer an equilibrium, since j has a profitable deviation to be assigned to $\mu_{PE}(i^*)$ because the rejection chain is stopped because $s \notin Q'_{PE_i}$ (from Lemma 1 and Lemma 2). Note that, this is true even if the application cost $c_{j,\mu PE(i^*)}$ is positive.

This is true as long as $\mathcal{Q}_T$ produces a stable assignment. Since the assignment suggested by DA is the stable assignment preferred by the students, we consider it in the proof. □

Proof of Corollary 2

Proof. Let us consider two cases:
Case 1: the obtained matching with true preferences is Pareto-efficient. In the presence of strictly positive cost, we have $\forall i \in \mathcal{I}, Q_i : \mu_I(i), i$ with $\mu_I(i)$ the school to which i is assigned using the true preferences under DA. Because the costs are small, student preferences are not modified. Student i would prefer to pay a positive cost (i.e. $c_{i,\mu_I(i)} = 1$) and be assigned to $\mu_I(i)$ rather than not be assigned at all.
Case 2: the obtained matching with true preferences is not Pareto-efficient. So, let us denote it by $\mu_{PE}(i)$ for every student $i \in \mathcal{I}$ the Pareto-efficient outcome. We know that for at least one student i, we have $\mu_I(i) \neq \mu_{PE}(i)$. In the presence of even very low costs for every school, we know that the preferred strategy is to apply to only one school. Then, $\forall i \in \mathcal{I}, Q_i : \mu_{PE}(i), i$. On the other hand, we know that this matching had not been obtained with DA. So there exists $i^*, j \in \mathcal{I}$ such that $\mu_{PE}(i^*) P_j \mu_{PE}(j)$ and $j \succ_{\mu_{PE}(i^*)} i^*$. So j has an interest in deviating from her strategy $Q_j : \mu_{PE}(i^*), j$ and the matching is no longer Pareto-efficient as i is no longer assigned. Consequently, it is not an equilibrium.

□

Proof of Theorem 3

Proof. We consider a strategy profile in which all students apply only to $\mu_I(i)$ and show that this is an equilibrium for every cost profile. Formally: for every $i \in \mathcal{I}, Q_i : \mu_I(i), i$. With this profile, the obtained matching is stable, which means that there is no student j such that $j, i^* \in \mathcal{I}$ and $\mu_I(i^*) P_j \mu_I(j)$ and $j \succ_{\mu_I(i^*)} i^*$. There is no profitable deviation for j to apply to $\mu_I(i^*)$ instead of $\mu_I(j)$. By definition of low costs, the preferences order is not modified. It is, therefore, straightforward to see that equilibrium is always possible. Thus, for every cost profile, there is at least one equilibrium. This reasoning is presented in Theorem 4.15 by Gale and Sotomayor in Roth and Sotomayor (1992). □

Proof of Proposition 6

Proof. We consider that schools have preferences for students. From the literature, we know that DA is strategy-proof in student-proposing for students. Assume that the schools' reported preference rankings are fixed and do not change during the analysis. Therefore, only the implementation of application costs is to be considered from a strategic point of view. We have presented four possible strategies. However, if the schools' payoff is not decreased with the number of applications to be considered, only two strategies can be studied. Let us denote Q_s, the strategy of school s. $Q_s^1 : c_s = 1$ means the school sets up an application cost for any students. $Q_s^0 : c_s = 0$, meaning that all students have zero application costs. Let us consider several cases:
Case 1: $\mu_I^{-1}(s) = \mu_{PE}^{-1}(s)$. s is indifferent between Q_s^1 and Q_s^0 no matter the strategy of the other schools.

Case 2: $\mu_I^{-1}(s) \neq \mu_{PE}^{-1}(s)$ and $\nexists t \in \mathcal{S}$ such that $\mu_I^{-1}(t) \neq \mu_{PE}^{-1}(t)$ and $Q_t = Q_t^1$. This means that the matching of s is different in the two equilibrium outcomes and that no school in the same situation has implemented a positive application cost. So we know that s prefers the matching $\mu_I^{-1}(s)$ to $\mu_{PE}^{-1}(s)$ and that μ_{PE} is not stable. Then, setting a positive cost prevents having a Pareto-efficient equilibrium from Theorem 2. With the deviation of a student involved in an interrupter, the rejection chain will be disrupted. However, the students assigned to s will be preferred to the students $\mu_{PE}^{-1}(s)$ by s. In that case, the strategy chosen by s will be Q_s^1.
Case 3: $\mu_I^{-1}(s) \neq \mu_{PE}^{-1}(s)$ and $\exists t \in \mathcal{S}$ such that $\mu_I^{-1}(t) \neq \mu_{PE}^{-1}(t)$ and $Q_t = Q_t^1$. If t is in the same rejection chain as s, then s is indifferent between Q_s^1 and Q_s^0. If t is not in the same rejection chain as s, then the strategy chosen by s will be Q_s^1.
In all situations, s weakly prefers Q_s^1 for every other school's strategies. □

References

Atila Abdulkadiroğlu and Tayfun Sönmez. School choice: A mechanism design approach. *American economic review*, 93(3):729–747, 2003.

Michel Balinski and Tayfun Sönmez. A tale of two mechanisms: student placement. *Journal of Economic theory*, 84(1):73–94, 1999.

Lars Ehlers and Jordi Massó. Matching markets under (in) complete information. *Journal of economic theory*, 157:295–314, 2015.

Haluk Ergin and Tayfun Sönmez. Games of school choice under the boston mechanism. *Journal of public Economics*, 90(1-2):215–237, 2006.

David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

Guillaume Haeringer and Flip Klijn. Constrained school choice. *Journal of Economic theory*, 144(5):1921–1947, 2009.

Yinghua He and Thierry Magnac. Application costs and congestion in matching markets. *The Economic Journal*, 132(648):2918–2950, 2022.

Onur Kesten. School choice with consent. *The Quarterly Journal of Economics*, 125(3):1297–1348, 2010.

Alvin E Roth. The economics of matching: Stability and incentives. *Mathematics of operations research*, 7(4):617–628, 1982.

Alvin E Roth and Marilda Sotomayor. Two-sided matching. *Handbook of game theory with economic applications*, 1:485–541, 1992.